Problem 2.3

Find the electric field a distance z above one end of a straight line segment of length L (Fig. 2.7) that carries a uniform line charge λ. Check that your formula is consistent with what you would expect for the case $z \gg L$.

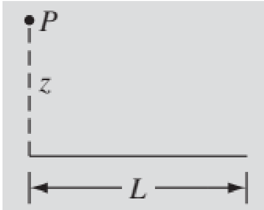
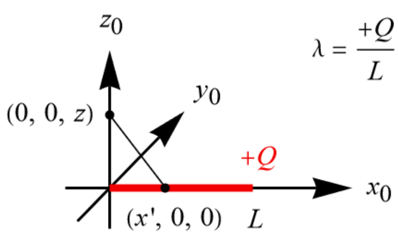


Fig. 2.7

Solution

Start by drawing a schematic for some point on the line segment.



The formula for the electric field from a continuous distribution of charge along a line is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,dl' = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)dl'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dl',$$

where the integral is taken over the line where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose on the line, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the line to where we want to know the electric field. The electric field at $\mathbf{r} = \langle 0,0,z\rangle$ is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_0^L \frac{\lambda}{\left[\sqrt{(0-x')^2+(0-0)^2+(z-0)^2}\right]^3}\left(\langle 0,0,z\rangle - \langle x',0,0\rangle\right)dx'$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{1}{(x'^2+z^2)^{3/2}}\langle -x',0,z\rangle\,dx'$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left\langle \int_0^L \frac{-1}{(x'^2+z^2)^{3/2}}\,x'\,dx', 0, z\int_0^L \frac{dx'}{(x'^2+z^2)^{3/2}}\right\rangle.$$

Make the following substitutions in these two integrals.

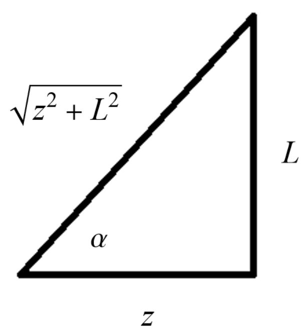
$$u = x'^2 + z^2 \qquad\qquad x' = z\tan\theta \quad \rightarrow \quad x'^2 + z^2 = z^2(\tan^2\theta + 1) = z^2\sec^2\theta$$

$$du = 2x'\,dx' \qquad\qquad dx' = z\sec^2\theta\,d\theta$$

Consequently,

$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0}\left\langle \int_{0^2+z^2}^{L^2+z^2} \frac{-1}{u^{3/2}}\left(\frac{du}{2}\right), 0, z\int_{\tan^{-1}\left(\frac{0}{z}\right)}^{\tan^{-1}\left(\frac{L}{z}\right)} \frac{z\sec^2\theta\,d\theta}{(z^2\sec^2\theta)^{3/2}} \right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle -\frac{1}{2}\int_{z^2}^{L^2+z^2} u^{-3/2}\,du, 0, z\int_0^{\tan^{-1}\left(\frac{L}{z}\right)} \frac{z\sec^2\theta\,d\theta}{z^3\sec^3\theta} \right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle -\frac{1}{2}(-2u^{-1/2})\Big|_{z^2}^{L^2+z^2}, 0, \frac{1}{z}\int_0^{\tan^{-1}\left(\frac{L}{z}\right)} \cos\theta\,d\theta \right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle (u^{-1/2})\Big|_{z^2}^{L^2+z^2}, 0, \frac{1}{z}(\sin\theta)\Big|_0^{\tan^{-1}\left(\frac{L}{z}\right)} \right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\left\langle \frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}, 0, \frac{1}{z}\sin\tan^{-1}\left(\frac{L}{z}\right) \right\rangle.
\end{aligned}$$

Draw the triangle implied by $\alpha = \tan^{-1}(L/z)$ and use it to determine $\sin\alpha$.



$$\sin\alpha = \frac{L}{\sqrt{z^2+L^2}}$$

Therefore, the electric field at $\mathbf{r} = \langle 0, 0, z\rangle$ is

$$\boxed{\mathbf{E} = \frac{\lambda}{4\pi\epsilon_0}\left\langle \frac{1}{\sqrt{L^2+z^2}} - \frac{1}{z}, 0, \frac{1}{z}\frac{L}{\sqrt{z^2+L^2}} \right\rangle.}$$

In order to see what happens if $z \gg L$, rewrite the formula so that each term is a ratio of L and z, z being in the denominator, and get rid of the square roots by using the binomial theorem.

$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0}\left\langle \frac{1}{z\sqrt{\frac{L^2}{z^2}+1}} - \frac{1}{z}, 0, \frac{1}{z}\frac{L}{z\sqrt{1+\frac{L^2}{z^2}}}\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \frac{1}{\sqrt{\frac{L^2}{z^2}+1}} - 1, 0, \frac{L}{z}\frac{1}{\sqrt{1+\frac{L^2}{z^2}}}\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \left(1+\frac{L^2}{z^2}\right)^{-1/2} - 1, 0, \frac{L}{z}\left(1+\frac{L^2}{z^2}\right)^{-1/2}\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k+1\right)}\left(\frac{L^2}{z^2}\right)^k - 1, 0, \frac{L}{z}\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k+1\right)}\left(\frac{L^2}{z^2}\right)^k\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}-k\right)}\left(\frac{L}{z}\right)^{2k} - 1, 0, \frac{L}{z}\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}-k\right)}\left(\frac{L}{z}\right)^{2k}\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}\left(\frac{L}{z}\right)^0 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{L}{z}\right)^2 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{3}{2}\right)}\left(\frac{L}{z}\right)^4 + \cdots\right] - 1, 0,\right. \\
&\qquad\qquad \left.\frac{L}{z}\left[\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}\left(\frac{L}{z}\right)^0 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{L}{z}\right)^2 + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{3}{2}\right)}\left(\frac{L}{z}\right)^4 + \cdots\right]\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle \left[1 - \frac{1}{2}\left(\frac{L^2}{z^2}\right) + \frac{3}{8}\left(\frac{L^4}{z^4}\right) - \cdots\right] - 1, 0, \frac{L}{z}\left[1 - \frac{1}{2}\left(\frac{L^2}{z^2}\right) + \frac{3}{8}\left(\frac{L^4}{z^4}\right) - \cdots\right]\right\rangle \\
&= \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle -\frac{L^2}{2z^2} + \frac{3L^4}{8z^4} - \cdots, 0, \frac{L}{z} - \frac{L^3}{2z^3} + \frac{3L^5}{8z^5} - \cdots\right\rangle
\end{aligned}$$

If $z \gg L$, then L/z is small, but L^2/z^2 and higher-order terms are so much smaller by comparison that they can be neglected.

$$\begin{aligned}
\mathbf{E} &\approx \frac{\lambda}{4\pi\epsilon_0}\frac{1}{z}\left\langle 0, 0, \frac{L}{z}\right\rangle \\
&\approx \frac{\lambda}{4\pi\epsilon_0}\frac{L}{z^2}\langle 0, 0, 1\rangle \\
&\approx \frac{\lambda}{4\pi\epsilon_0}\frac{L}{z^2}\hat{\mathbf{z}} \\
&\approx \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\hat{\mathbf{z}}
\end{aligned}$$

The lesson is that far away from the line segment the electric field is the same as if it were a point charge.